

Some examples of classical coboundary Lie bialgebras with coboundary duals

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Abstract

Some examples are given of finite dimensional Lie bialgebras whose brackets and cobrackets are determined by pairs of r -matrices.

The aim of this Letter is to give some low-dimensional examples of classical coboundary Lie bialgebras [1, 2] with coboundary duals. Since such structures can be specified (up to automorphisms) by pairs of r -matrices, so it is natural to call them *bi- r -matrix bialgebras* (BrB). There are some reasons to study BrB. Assuming that both Lie bialgebras of a dual pair are coboundary, we impose additional constraints, which can facilitate the search of new classical r -matrices connected with nonsemisimple Lie algebras. Recall that many Lie algebras of physical interest are nonsemisimple ones, but up to now there is detailed classification of r -matrices only for the complex simple Lie algebras [3]. The presence of a pair of r -matrices is useful for practical quantization of Lie bialgebras [1] permitting more symmetrical treatment of quantum algebras and groups. However, most interesting applications of BrB are possible in the theory of bihamiltonian dynamical systems [4, 5]. In this case the presence of a pair of r -matrices allows us to define the pair of dynamical systems on the space which is the space of the original Lie algebra canonically identified with its dual space [6].

Now recall some basic definitions [1, 2]. Let L be a finite-dimensional Lie algebra and L^* the dual space of L in respect to a nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ on $L^* \times L$. The Lie algebra L is called a bialgebra, if there exist a map $\delta : L \rightarrow L \otimes L$ which is an 1-cocycle

$$\delta([x, y]) = [\delta(x), y \otimes 1 + 1 \otimes y] + [x \otimes 1 + 1 \otimes x, \delta(y)], \quad x, y \in L \quad (1)$$

and which defines on L^* a Lie algebra structure $[\cdot, \cdot]_* : L^* \otimes L^* \rightarrow L^*$ by the following relation

$$\langle [\xi, \eta]_*, x \rangle = \langle \xi \otimes \eta, \delta(x) \rangle, \quad x \in L, \quad \xi, \eta \in L^*. \quad (2)$$

If one puts

$$\delta(x) = [x \otimes 1 + 1 \otimes x, r], \quad r = r^{mn} e_m \otimes e_n \in L \otimes L \quad (3)$$

then the 1-cocycle condition (1) is fulfilled identically. In this case the Lie bialgebra L is called a coboundary one. It can be shown [2] that the Jacobi identity for the elements of L^* is equivalent to the following equation

$$[x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x, [r, r]_s] = 0, \quad x \in L. \quad (4)$$

The Schouten brackets $[r, r]_s$ in the above formula are defined by

$$[r, r]_s \equiv [r_{12}, r_{13} + r_{23}] + [r_{13}, r_{23}]$$

where, as usual, $r_{12} = r^{mn} e_m \otimes e_n \otimes 1$, etc. The simplest way to satisfy the relation (4) is to put

$$[r, r]_s = 0.$$

This is the classical Yang-Baxter equation and its solutions for the complex simple Lie algebras are classified in the mentioned work [3]. If some r -matrix satisfies CYBE, then the related bialgebra (as well as this r -matrix itself) is called a quasitriangular bialgebra. If, in addition, this r -matrix satisfies the unitarity condition

$$\sigma \circ r = -r$$

where σ is the flip operator $\sigma(x \otimes y) = y \otimes x$, then the bialgebra is called a triangular one. When we leaving out \mathbf{Z}_2 -graded Lie bialgebras, we can restrict ourselves by unitary r -matrices because brackets $[\cdot, \cdot]_*$ are defined only by the antisymmetric part of r .

Now extend slightly the above definition of bi- r -matrix bialgebras. Let L be a coboundary Lie bialgebra with an 1-cocycle (3) and L^* be its dual space endowed with a structure of coboundary Lie bialgebra as well. Let δ^* be a corresponding 1-cocycle defined by some r -matrix r^*

$$\delta^*(\xi) = [\xi \otimes 1 + 1 \otimes \xi, r^*]_*, \quad \xi \in L^*, \quad r^* \in L^* \otimes L^*.$$

We will call the pair (L, L^*) a bi- r -matrix bialgebra if the Lie brackets $[\cdot, \cdot]'$ on L defined by δ^*

$$\langle \delta^*(\xi), x \otimes y \rangle = \langle \xi, [x, y]' \rangle, \quad x, y \in L, \quad \xi \in L^*$$

are equivalent to the original ones

$$[x, y]' = S^{-1}[Sx, Sy], \quad x, y \in L, \quad S \in \text{Aut}(L).$$

Because of the equal status of the algebras L and L^* in our definition, we will call BrB under consideration a double name in what follows.

The simplest and well known example of a self-dual BrB gives the case of the two dimensional non-Abelian Lie algebra with the generators e_1, e_2 [1]. Let the normalized commutation relation of this algebra be

$$[e_1, e_2] = e_1. \tag{5}$$

The only possible triangular r -matrix

$$r = e_1 \wedge e_2$$

produces by (2), (3) the relation among the generators e^1, e^2 of L^* (we use the form $\langle e^i, e_j \rangle = \delta_j^i$)

$$[e^1, e^2]_* = e^2.$$

The r -matrix $r^* \in L^* \wedge L^*$

$$r^* = -e^1 \wedge e^2$$

induces on L the original brackets (5).

Consider the case of a three-dimensional complex Lie algebra L . It is well known that the commutation relations among its generators $\{e_i\}$, $i = 1, 2, 3$ can be reduced by a linear transformation to the following form

$$[e_1, e_2] = ae_2 + be_3, \quad [e_2, e_3] = ce_1, \quad [e_1, e_3] = fe_2 + ae_3. \quad (6)$$

The Jacobi identity restricts the values of the structure constants a, b, c, f by the condition $ac = 0$. At first consider the case $a = 0, c \neq 0$. An arbitrary unitary r -matrix

$$r = r^{12}e_1 \wedge e_2 + r^{13}e_1 \wedge e_3 + r^{23}e_2 \wedge e_3 \quad (7)$$

gives the relations among the generators $\{e^i\}$, $i = 1, 2, 3$ of the dual Lie algebra L^*

$$[e^1, e^2]_* = fr^{13}e^1 - cr^{23}e^2, \quad [e^1, e^3]_* = br^{12}e^1 - cr^{23}e^3, \quad [e^2, e^3]_* = br^{12}e^2 - fr^{13}e^3 \quad (8)$$

which satisfy the Jacobi identity. Note that the structure constants and r -matrix elements can be mutually absorbed ($fr^{13} \rightarrow r^{13}$ or $fr^{13} \rightarrow f$, etc.). For the Schouten brackets one has

$$[r, r]_s = (b(r^{12})^2 - f(r^{13})^2 + c(r^{23})^2) e_1 \wedge e_2 \wedge e_3. \quad (9)$$

It is not difficult to show that any choice of the structure constants b, c, f and the r -matrix elements r^{12}, r^{13}, r^{23} in the non-Abelian case leads to commutators on L^* which are equivalent to the following ones

$$[\tilde{e}^i, \tilde{e}^j]_* = \tilde{e}^j, \quad [\tilde{e}^j, \tilde{e}^k]_* = 0, \quad [\tilde{e}^i, \tilde{e}^k]_* = \tilde{e}^k \quad (10)$$

where (i, j, k) is some transposition of $(1, 2, 3)$. The Lie algebra with the above commutation relations (we will denominate it $j(3)$) appeared, in particular, in the semiclassical limit of the matrix quantum group $SL_q(2)$ and the Jordanian group $SL_h(2)$ (see, for example, [7, 8]).

Taking into account the above results, consider the special case $a = 1, b = c = f = 0$. For these structure constants the relations (6) take the form

$$[e_1, e_2] = e_2, \quad [e_2, e_3] = 0, \quad [e_1, e_3] = e_3 \quad (11)$$

and the Schouten brackets for an arbitrary r -matrix (7) identically vanish $[r, r]_s = 0$. The commutation relations obtained from (7), (11) by (2), (3)

$$[e^1, e^2]_* = r^{12}e^1, \quad [e^1, e^3]_* = r^{13}e^1, \quad [e^2, e^3]_* = 2r^{23}e^1 - r^{13}e^2 + r^{12}e^3. \quad (12)$$

satisfy the Jacobi identity.

Summarizing the above formulae (6) – (12), list in the Table 1 (placed in the Appendix) the most interesting from the physical point of view three-dimensional bi- r -matrix bialgebras: the

Heisenberg-Weyl algebra $h(3)$ ($a = b = f = 0; c = 1$) — $j(3)$, the Euclidean $e(2)$ ($a = b = 0, f = -1; c = 1$) — $j(3)$ and the Poincarè $p(2)$ ($a = b = 0, f = -1; c = -1$) — $j(3)$. For the sake of clearness we consider the Lie algebras of the groups of the Euclidean and pseudoeuclidean planes separately in spite of these cases are distinguished unessentially in the structure constants and coefficients of r -matrices. Remark, that some r -matrices defining the brackets $[\cdot, \cdot]_*$ and presented in the Table 1 are well known. For example, $r = -e_2 \wedge e_3$ was considered in the papers [9, 10, 11, 12], and $r = e_1 \wedge e_3 + ie_2 \wedge e_3$ in [13].

Both of the dual algebras for $e(2)$ and $p(2)$, as was mentioned above, are equivalent to the one with the commutation relations (10). It can be displayed explicitly. For example, in the case $p(2)$, using the r -matrix $r = e_2 \wedge e_3$ instead of $r = e_1 \wedge e_3 + e_2 \wedge e_3$, one obtains the relations

$$[e^1, e^2]_* = e^2, \quad [e^2, e^3]_* = 0, \quad [e^1, e^3]_* = e^3$$

directly. By the r -matrix $r^* = e^1 \wedge e^3$ one has

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = -e_2.$$

After the transformation

$$S : e_1 \mapsto \tilde{e}_1 = e_1 + e_2, \quad e_2 \mapsto \tilde{e}_2 = e_1 - e_2, \quad e_3 \mapsto \tilde{e}_3 = -e_3,$$

we restore the original $p(2)$ commutation relations. Let us stress here once more that all of the considered three-dimensional bialgebras $h(3), e(2), p(2)$ and the bialgebra $sl(2, C)$ have the same Lie algebra as a dual counterpart. This fact can be used for their unified multiparameter quantization [14].

Until now we used the unitary r -matrices because only their antisymmetric parts $r^a = 1/2(r - \sigma \circ r)$ define the brackets for the dual Lie algebras. When we turn to the case of Lie superalgebras the symmetric parts $r^s = 1/2(r + \sigma \circ r)$ play the important role. We placed in the Table 1 the three-dimensional bi- r -matrix superbialgebra in which the dual pair is the Poincarè algebra $p(2)$ and the anticommutator algebra $c(3)$. In this case the equivalence of the brackets $[\cdot, \cdot]'$ given by the r -matrix $r^* = e^1 \wedge e^2$ and the original $p(2)$ — brackets is displayed by the transformation $S : e^3 \mapsto \tilde{e}^3, \quad -e^1 + e^2 \mapsto \tilde{e}^1, \quad e^1 + e^2 \mapsto \tilde{e}^2$.

In fact, in the Table 1 are listed all possible three-dimensional BrB and no one of them is self-dual. However, the extending of the Heisenberg-Weyl algebra $h(3)$ gives the self-dual example. This is so-called (1+1) extended Galilei algebra. This algebra was studied in detailed in the work [15] and the formulae placed in the Table 1 are taken from there.

It is not difficult to generalize the above low-dimensional examples to the n -dimensional cases. To facilitate the search of new BrB it is useful to take into account the following assertions. Let L be a n -dimensional complex Lie algebra, and I its subalgebra. As a space L can be represented by the sum $L = I \oplus K$. Denote by I' and K' the subspaces of the dual space $L^* = I' \oplus K'$ which are orthogonal to K and I respectively: $\langle I', K \rangle = \langle K', I \rangle = 0$. Let us list simple properties which are the direct consequences of the formula (3) provided that $\delta(x) = [x \otimes 1 + 1 \otimes x, r]$, $x \in L$, $r \in L \otimes L$ defines on L the structure of a coboundary Lie bialgebra.

- If I is a subalgebra of L , $r \in L \otimes L \oplus K \otimes K$, then K' is an subalgebra of L^* .

- If I is an ideal, then K' is a subalgebra of L^* .
- If I is an ideal, $r \in I \otimes I$, then K' is a centre of L^* .
- If I is a centre, then K' is the commutant of L^* .
- If I is a commutative ideal, $r \in I \otimes I$, then K' is both the commutant and the centre of L^* .
- If I is both a commutant and a centre, $r \in K \otimes K$, then K' the commutative commutant.

The last two points allow directly generalize the Heisenberg-Weyl bi- r -matrix bialgebra $h(3) - j(3)$ to the n -dimensional case (see Table 1).

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A Appendix

Table 1. Some bi- r -matrix bialgebras.

L	r	L^*	r^*
The Heisenberg-Weyl algebra $h(3) - j(3)$			
$[e_1, e_2] = 0$ $[e_2, e_3] = e_1$ $[e_1, e_3] = 0$	$-e_2 \wedge e_3$ [9, 10, 11, 12]	$[e^1, e^2]_* = e^2$ $[e^2, e^3]_* = 0$ $[e^1, e^3]_* = e^3$	$\frac{1}{2}e^2 \wedge e^3$
The algebra of the Euclidean plane $e(2) - j(3)$			
$[e_1, e_2] = 0$ $[e_2, e_3] = e_1$ $[e_3, e_1] = e_2$	$e_1 \wedge e_3 + ie_2 \wedge e_3$ [13]	$[e^1, e^2]_* = -e^1 - ie^2$ $[e^2, e^3]_* = e^3$ $[e^1, e^3]_* = -ie^3$	$-\frac{1}{2}(e^1 \wedge e^3 - ie^2 \wedge e^3)$
The algebra of the pseudoeuclidean plane $p(2) - j(3)$			
$[e_1, e_2] = 0$ $[e_2, e_3] = -e_1$ $[e_3, e_1] = e_2$	$e_1 \wedge e_3 + e_2 \wedge e_3$	$[e^1, e^2]_* = -e^1 + e^2$ $[e^2, e^3]_* = e^3$ $[e^1, e^3]_* = e^3$	$-\frac{1}{2}(e^1 \wedge e^3 + e^2 \wedge e^3)$
The algebra of the pseudoeuclidean plane $p(2)$ — The Clifford algebra $c(3)$			
$[e_1, e_2] = 0$ $[e_3, e_2] = e_1$ $[e_3, e_1] = e_2$	$1/2(e_1 \otimes e_1 + e_2 \otimes e_2)$	$\{e^1, e^2\}_* = e^3$ $[e^1, e^3]_* = 0$ $[e^2, e^3]_* = 0$	$e^1 \wedge e^2$
Self-dual extended $(1+1)$ Galilei algebra			
$[e_3, e_1] = e_2$ $[e_3, e_2] = e_4$ $[e_1, e_2] = 0$ $[e_4, \cdot] = 0$	$e_1 \wedge e_2 - e_3 \wedge e_4$ [15]	$[e^2, e^4]_* = e^1$ $[e^1, e^4]_* = e^3$ $[e^1, e^2]_* = 0$ $[e^3, \cdot]_* = 0$	$-(e^1 \wedge e^2 - e^3 \wedge e^4)[15]$
The Heisenberg-Weyl algebra $h(3n) - j(3n)$			
$[e_{2i}, e_{3j}] = \delta_{ij}e_1$ $[e_{2i}, e_{2j}] = 0$ $[e_{3i}, e_{3j}] = 0$ $i, j = 1, \dots, n$ $[e_1, \cdot] = 0$	$\sum_{i=1}^n a^i e_{2i} \wedge e_{3i}$ [9]	$[e^{2i}, e^{1j}]_* = a^i e^{2i}$ $[e^{3i}, e^{1j}]_* = a^i e^{3i}$ $[e^{2i}, e^{3j}]_* = 0$ $i, j = 1, \dots, n$	$-\frac{1}{2} \sum_{i=1}^n (a^i)^{-1} e^{2i} \wedge e^{3i}$

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